The Game of Chomps and the Angel Game

ES 214 Discrete Mathematics

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Game Theoretic Terminologies

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- **Strategy:** In game theory, a player's strategy is any of the options which he or she chooses in a setting where the outcome depends not only on their own actions but on the actions of others. A player's strategy will determine the action which the player will take at any stage of the game.

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- **Perfect Information:** A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.
- **Strategic Form:** A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies.

Game of Chomps

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The Game of Chomps
Game Description

Chomp is a game played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned, as shown in figure below.

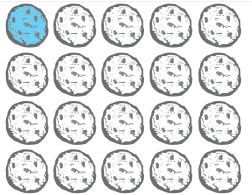


Figure 1: Setting of the Game of Chomp

The Game of Chomps
Game Rules

The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below it. The loser is the player who has no choice but to eat the poisoned cookie. The figure illustrating valid moves is as shown.

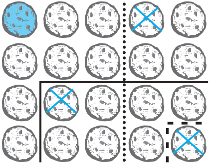


Figure 2: Illustration of a Valid Move

Existence of Winning Strategy

Strategy Stealing Argument

Lemma

There exists a winning strategy for the first player.

Proof

We will give a nonconstructive existence proof of a winning strategy for the first layer. That is, we will show that the first player always has a winning strategy without explicitly describing the moves this player must follow.

First, note that the game ends and cannot finish in a draw because with each move at least one cookie is eaten, so after no more than m n moves the game ends, where the initial grid is m n.

Now we will use the **strategy-stealing argument** to prove this lemma. Suppose that the first player begins the game by eating just the cookie in the bottom right corner. There are two possibilities, this is the first move of a winning strategy for the first player, or the second player can make a move that is the first move of a winning strategy for the second player. In this second case, instead of eating just the cookie in the bottom right corner, the first player could have made the same move that the second player made as the first move of a winning strategy (and then continued to follow that winning strategy).
This would guarantee a win for the first player. Note that we showed that a winning strategy exists, but we did not specify an actual winning strategy. Consequently, the proof is a nonconstructive existence proof.

Winning Strategies for some Special Cases

Winning Strategy for a $2 \times n$ board

Lemma

The first player always has a winning strategy on a $2 \quad n$ board.

Proof

Let $P(n)$ be "The first player has a winning strategy for the game of Chomp for a $2 \quad n$ board". For the base case, we have $2 \quad 1$ board. The first player eats the non poisoned cookie and then the second player has to eat the poisoned cookie, which means that the first player wins. Thus, $P(1)$ is true.

We assume that $P(i)$ is true for $1 \quad i \quad k$. We need to prove that $P(k+1)$ is true.

The first move of the first player is to eat the cookie in the bottom right corner. When the opponent needs to make a move and the opponent eats the poisoned cookie, then the first player wins. When the opponent makes a move and does not eat the poisoned cookie, then at some point the opponent needs to pick a cookie in the first row or column (Note that as long as the opponent doesn't, the first player picks the cookie above and one position to the right of the opponent's cookie, such that we always obtain a rectangle with the bottom right corner missing)
If the player picked a cookie in the first column, then the first player wins by selecting the second cookie in the first row. If the player picked a cookie in the first row, then the remaining cookies form 2 j rectangle (with $j < k+1$). Since $P(j)$ is true, the first player then can win the game. Hence $P(k+1)$ is true.

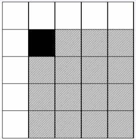
Winning Strategy for a n n board.

Lemma

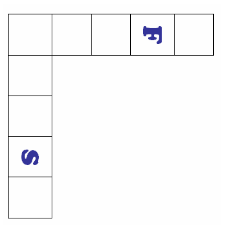
The first player always has a winning strategy on n n board

Proof

Let the Chomp board have n rows and n columns. We claim that the first player can win the game by making the first move to leave just the top row and leftmost column.



Then what the first player needs to do is to "steal" the second player's strategy by using the symmetry of the L shaped board.



The first player then steals the strategy of the second player.
The existence of the winning strategy for the first player can easily be shown by strong induction.

Generalisations of Chomp

Three-dimensional Chomp

Three-dimensional Chomp has an initial chocolate bar of a cuboid of blocks indexed as (i, j, k). A move is to take a block together with any block all of whose indices are greater or equal to the corresponding index of the chosen block.

Numerical Version of Chomp

An initial natural number is given, and players alternate choosing positive divisors of the initial number, but may not choose 1 or a multiple of a previously chosen divisor. This models n-dimensional Chomp, where the initial natural number has n prime factors and dimensions of the Chomp board are given by the exponents of the primes in its prime factorization.

Generalisations of Chomp

Ordinal Chomp

Ordinal Chomp is played on an infinite board with some of its dimensions ordinal numbers: for example a $2 \quad (\omega + 4)$ bar. A move is to pick any block and remove all blocks with both indices greater than or equal the corresponding indices of the chosen block. The case of $\omega \quad \omega \quad \omega$ Chomp is a notable open problem; a $100 reward has been offered for finding a winning first move.

Open Question

Describe a winning strategy for that Chomp that applies for all rectangular grids of size m n by describing the moves that the first player should follow.

Angel Game

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Angel Game
Game Description

- **Proposed by John Conway**;
- **An infinite chessboard**;
- **The Devil** eats a square on every turn;
- **A k-Angel** (Angel of power k) sits on a square and on each turn flies to an uneaten square within l_1 distance k.

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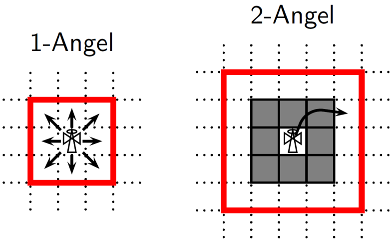

The Aim of the Game

- **The Devil** wants to trap the Angel (i.e. every square within reach of the Angel is eaten);
- **The Angel** wants to live forever.

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The angel-devil game is determined. That is, either the angel or the devil has a winning strategy.

Proof

- Suppose no winning strategy for Devil.
- Then Angel can always make a move that gives no winning strategy for the Devil in the new situation either.
- This gives a strategy where the Angel survives into infinity.
  $\Rightarrow$ A winning strategy for the Angel.

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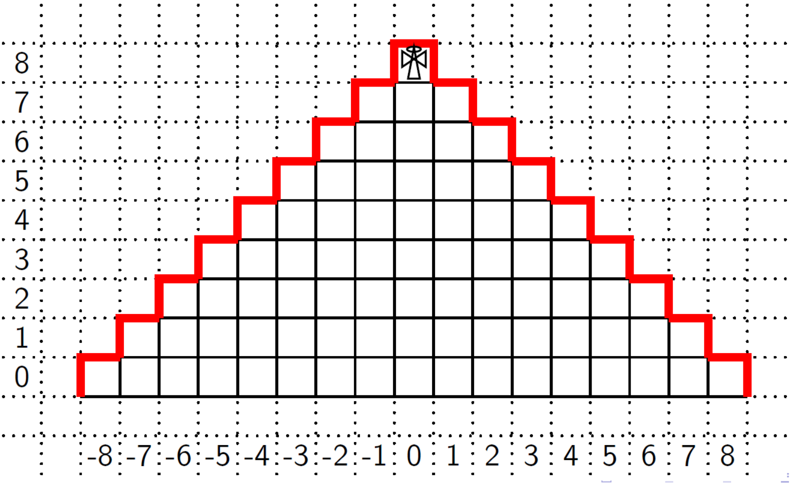
Catching a 1-Fool

Definition

A Fool is an Angel who is required always strictly to increase his y coordinate. So a Fool can make precisely those Angel moves from (x, y) to (X, Y) for which $Y > y$.

Catching a 1-Fool



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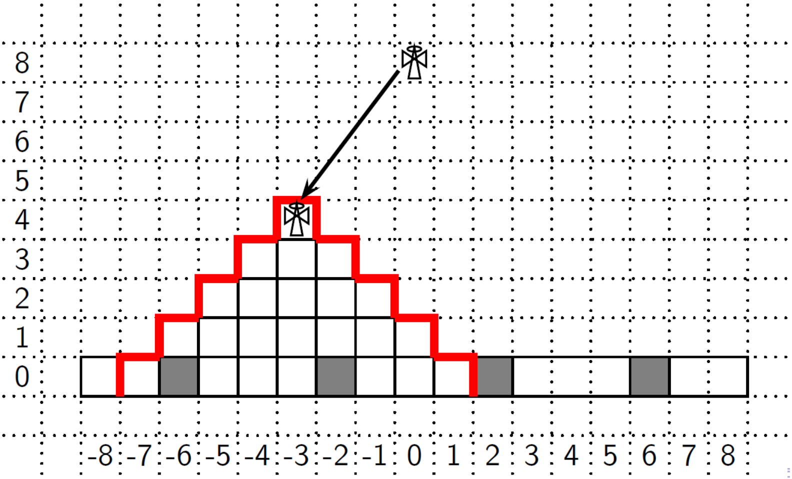

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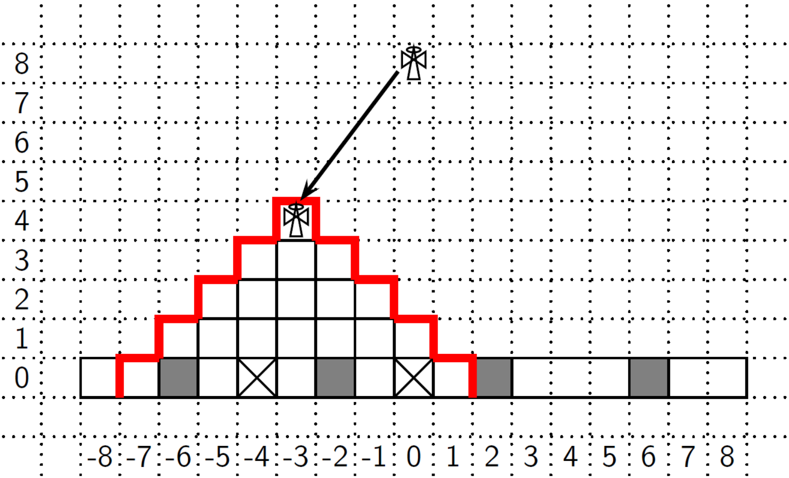

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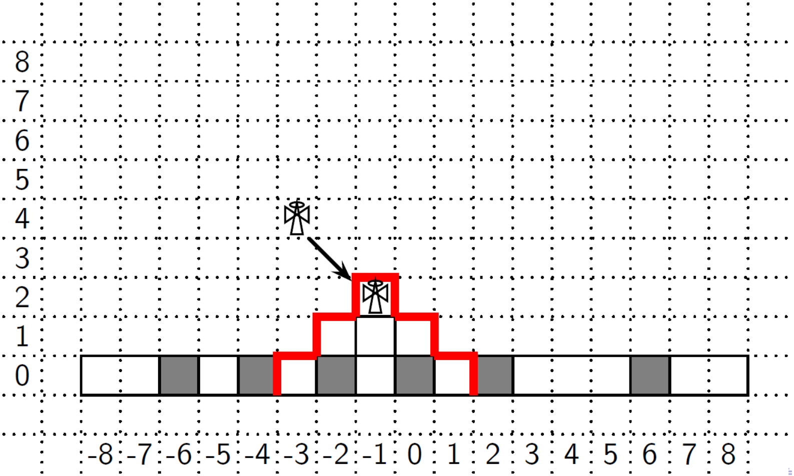

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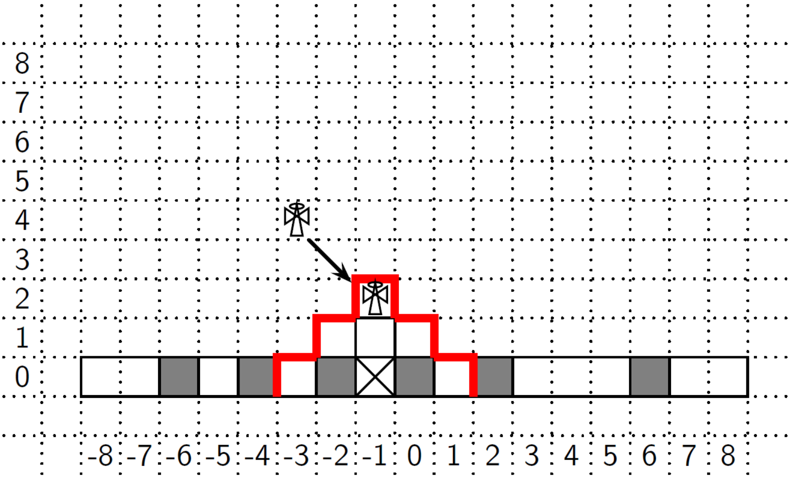

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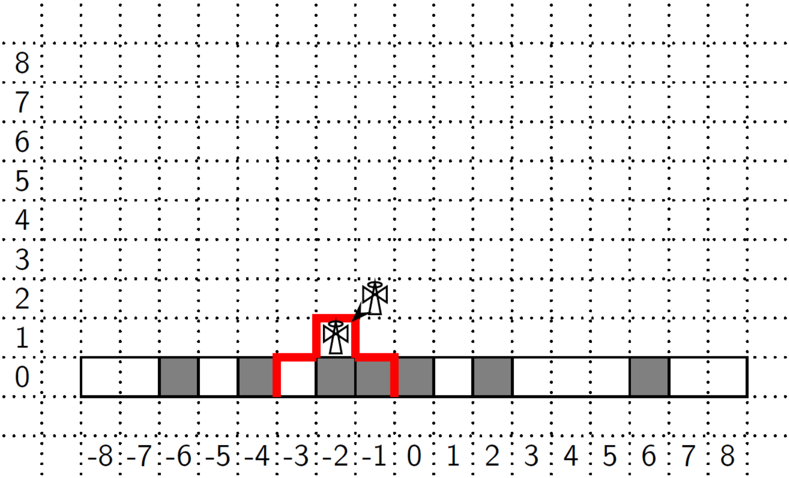

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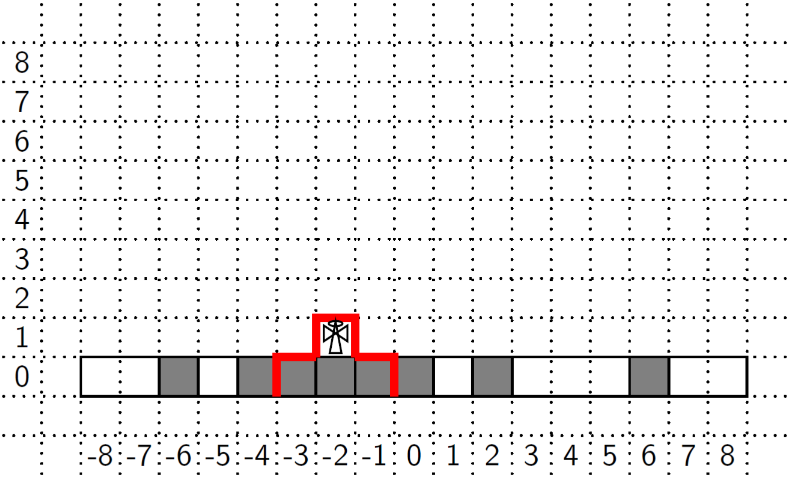

The Angel Wins!

In 2006, four independent proofs that some Angel wins:

1. Peter Gács: Angel of some power wins
2. Brian Bowditch: 4-Angel wins
3. András Máthé: 2-Angel wins
4. Oddvar Kloster: 2-Angel wins

We will discuss Kolster's Proof in the subsequent slides.

Oddvar Kolster Proof

- Oddvar Kolster presented a proof that showed the existence of a winning strategy for an Angel of Power 2.

Oddvar Kolster Proof

- The Angel walks along the right of a directed path.
- Initially the path a vertical line, with the angel directly to the right of it.

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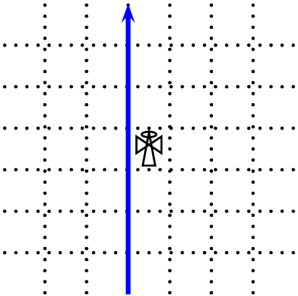

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- Each turn, the Angel tries to cover as much path as possible.
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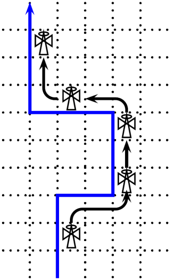
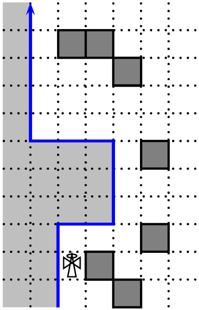


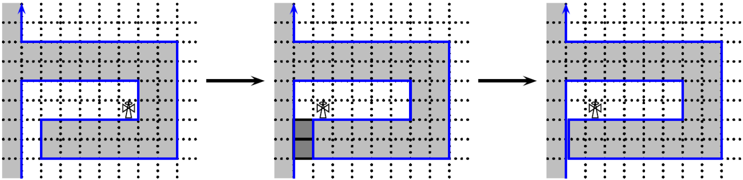
Oddvar Kolster Proof

- Squares are either free, or blocked.
- Squares to the left of the path are called evaded.



Oddvar Kolster Proof

- The Angel has to be careful how he adjusts the path, to avoid being trapped:



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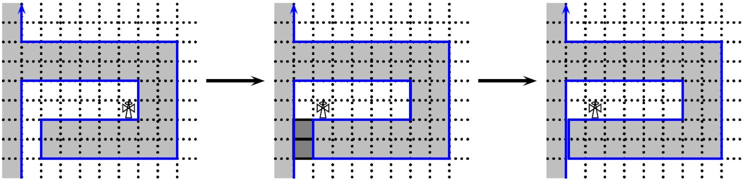

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Odvar Kolster Proof

- We assign an integer length L_k to a path k, namely the number of extra segments it has compared to the original path.
- We define $E_k(s)$ to be the number of squares, eaten before stage s, that are evaded by k. At stage s:

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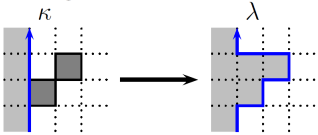


$$L_\lambda = L_\kappa + 4$$
$$E_\lambda(s) = E_\kappa(s) + 2$$

Oddvar Kolster Proof

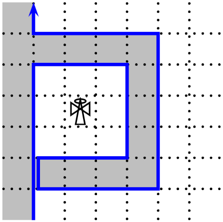
Strategy

Strategy

Among all the descendants of the previous path, select the path k with maximal $\mathbf{2}E_k(s) \quad L_k$. If there exists more than one such path, then select the one with maximal $E_k(s)$.

Oddvar Kolster Proof

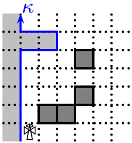
Can a square, just right of a future segment of the path, be evaded? If so, the Angel might be trapped:



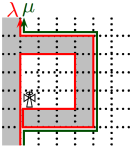
We will prove that the Angel, following our strategy, would have spotted this potential trap in time, and would have planned its path around the trap.

Oddvar Kolster Proof

Stage s:



Stage t:



Angel passes segments more than twice faster than the Devil can build:

$$L_\lambda \quad L_\mu > 2(E_\mu(t) \quad E_\mu(s))$$

Hence,

$$2E_\mu(s) \quad L_\mu > 2E_\mu(t) \quad L_\lambda$$

Oddvar Kolster Proof

$$2E_{\mu}(s) \quad L_{\mu} \quad 2E_{\lambda}(t) \quad L_{\lambda}$$

$$2E_{\mu}(s) \quad L_{\mu} \quad 2E_{k}(s) \quad L_{k}$$

Hence, at stage s, the Angel would have preferred path μ over path k.

Open Question

- **Can an Angel of power k win for some k?**. This problem was proposed by John Conway and it is still an open problem. There is a $100 prize for a winning Angel proof and $1000 prize for a winning Devil proof.

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THANK YOU