INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR

ES 214 WRITING PROJECT
DISCRETE MATHEMATICS

Instructor: Prof. Neeldhara Misra

SEM II 2018-19

The Game of Chomps and the Angel Game

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Abstract

This article basically describes about two well known combinatorial games - Game of Chomps and the Angel Game. We begin by describing some frequent terms used in combinatorial game theory. Then we delve into winning strategies for specific boards for the Game of Chomps and then generalize it. The winning strategy for a general $m \times n$ board is still an open question. Then we discuss the Angel Game proposed by John Conway. We see that any k-Fool can be caught by the Devil. We finally see Kloster's proof that a winning strategy exists for a 2-Angel. Whether for a k-Angel, which one - the Devil or the Angel has a winning strategy is still an open question.

Contents

1 Introduction

Combinatorial Game Theory (CGT) is a branch of mathematics and theoretical computer science that typically studies sequential games with perfect information. Study has been largely confined to two-player games that have a position in which the players take turns changing in defined ways or moves to achieve a defined winning condition. CGT has not traditionally studied games of chance or those that use imperfect or incomplete information, favoring games that offer perfect information in which the state of the game and the set of available moves is always known by both players. However, as mathematical techniques advance, the types of game that can be mathematically analyzed expands, thus the boundaries of the field are ever changing

Combinatorial games include well-known games such as chess, checkers, and Go, which are regarded as non-trivial, and tic-tac-toe, which is considered as trivial in the sense of being "easy to solve". Some combinatorial games may also have an unbounded playing area, such as infinite chess. In CGT, the moves in these and other games are represented as a game tree.

CGT has a different emphasis than“traditional" or “economic" game theory, which was initially developed to study games with simple combinatorial structure, but with elements of chance. Essentially, CGT has contributed new methods for analyzing game trees, for example using surreal numbers, which are a subclass of all two-player perfect-information games. The type of games studied by CGT is also of interest in artificial intelligence, particularly for automated planning and scheduling. In CGT there has been less emphasis on refining practical search algorithms, but more emphasis on descriptive theoretical results (such as measures of game complexity or proofs of optimal solution existence without necessarily specifying an algorithm, such as the strategy-stealing argument).

We will define some important terminologies related to game theory on the next page:

2 Game Theoretic Terminologies

2.1 Common Knowledge

- A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

2.2 Game tree

- In game theory, a game tree is a directed graph whose nodes are positions in a game and whose edges are moves. The complete game tree for a game is the game tree starting at the initial position and containing all possible moves from each position; the complete tree is the same tree as that obtained from the extensive-form game representation, the definition of which follows.

2.3 Extensive Game

- An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

2.4 Payoff

- A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

2.5 Strategy

- In game theory, a player's strategy is any of the options which he or she chooses in a setting where the outcome depends not only on their own actions but on the actions of others. A player's strategy will determine the action which the player will take at any stage of the game.

2.6 Strategic Move

- A strategic move in game theory is an action taken by a player outside the defined actions of the game in order to gain a strategic advantage and increase one's payoff.

2.7 Dominating Strategy

- A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing.

2.8 Perfect Information

- A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.

2.9 Strategic Form

- A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

2.10 Zero sum game

- A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed

3 The Game of Chomps

3.1 Game Description

Chomp is a game played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned, as shown in figure below.

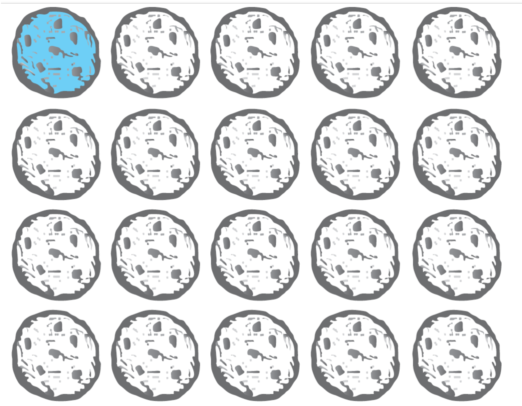

Figure 1 Setting of the Game of Chomp

The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below it. The loser is the player who has no choice but to eat the poisoned cookie.

The figure illustrating valid moves is as shown.

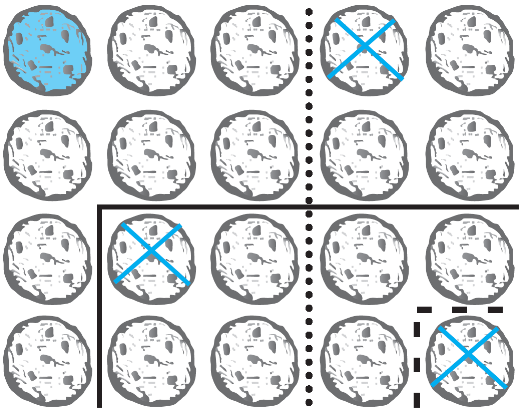

Figure 2 Illustration of a Valid Move

Chomp is a special case of a **poset game** where the partially ordered set on which the game is played is a product of total orders with the minimal element (poisonous block) removed.

The description of the game in the context of poset (partially ordered set) is as follows:

Given a partially ordered set (P, $<$), let

$$P_x = P - \{a|a \geq x\}$$

denote the poset formed by removing x from P.

A poset game on P, played between two players conventionally named Alice and Bob, is as follows:

- Alice chooses a point x $\in$ P; thus replacing P with P_x, and then passes the turn to Bob who plays on P_x, and passes the turn to Alice.
- A player loses if it is his/her turn and there are no points to choose.

3.2 Existence of a Winning Strategy - The Strategy-Stealing Argument

LEMMA: There exists a winning strategy for the first player.

PROOF: We will give a nonconstructive existence proof of a winning strategy for the first layer. That is, we will show that the first player always has a winning strategy without explicitly describing the moves this player must follow.

First, note that the game ends and cannot finish in a draw because with each move at least one cookie is eaten, so after no more than $m \times n$ moves the game ends, where the initial grid is $m \times n$. Now we will use the **strategy-stealing argument** to prove this lemma.

Now, suppose that the first player begins the game by eating just the cookie in the bottom right corner. There are two possibilities, this is the first move of a winning strategy for the first player, or the second player can make a move that is the first move of a winning strategy for the second player. In this second case, instead of eating just the cookie in the bottom right corner, the first player could have made the same move that the second player made as the first move of a winning strategy (and then continued to follow that winning strategy).

This would guarantee a win for the first player. Note that we showed that a winning strategy exists, but we did not specify an actual winning strategy. Consequently, the proof is a nonconstructive existence proof.

■

3.3 Description of Winning Strategies for some Special Cases

3.3.1 Winning Strategy for a $2 \times n$ board

Lemma: *The first player always has a winning strategy on a $2 \times n$ board.*

Proof: Let $P(n)$ be "The first player has a winning strategy for the game of Chomp for a $2 \times n$ board". For the base case, we have 2×1 board. The first player eats the non poisoned cookie and then the second player has no choice but to eat the poisoned cookie, which means that the first player wins. Thus, $P(1)$ is true.

We assume that $P(1)$, $P(2)$, ... , $P(k)$ is true. We need to prove that $P(k+1)$ is true. The first move of the first player is to eat the cookie in the bottom right corner. When the opponent needs to make a move and the opponent eats the poisoned cookie, then the first player wins. When the opponent makes a move and does not eat the poisoned cookie, then at some point the opponent needs to pick a cookie in the first row or column (Note that as long as the opponent doesn't, the first player picks the cookie above and one position to the right of the opponent's cookie, such that we always obtain a rectangle with the bottom right corner missing)

If the player picked a cookie in the first column, then the first player wins by selecting the second cookie in the first row. If the player picked a cookie in the first row, then the remaining cookies form $2 \times j$ rectangle (with $j < k+1$). Since $P(j)$ is true, the first player then can win the game. Hence $P(k+1)$ is true. ∎

3.3.2 Winning Strategy for a $n \times n$ board.

Lemma: *The first player always has a winning strategy on $n \times n$ board.*

Proof: Let the Chomp board have n rows and n columns. We claim that the first player can win the game by making the first move to leave just the top row and leftmost column. Here is the move which ensures this. The first player first selects the following black box and then removes it as well as those grey boxes.

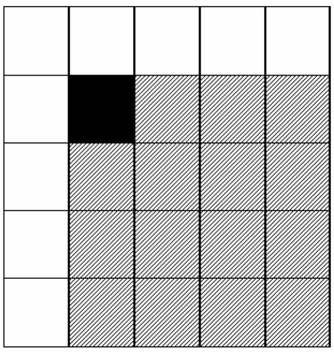

Figure 3 The first player chooses the black square thus leaving an L shaped board which then invokes a strategy stealing argument

Then what the first player needs to do is to "steal" the second player's strategy by using the symmetry of the L shaped board.

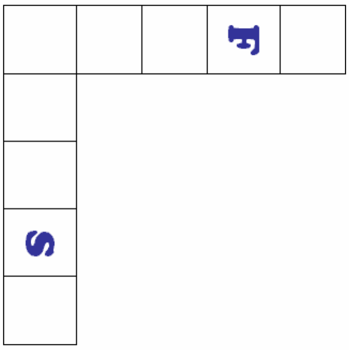

Figure 4 The first player steals the strategy of the second player

Let $P(n)$ be the statement that if a player has presented his opponent with a Chomp configuration consisting of just n cookies in the top row and n cookies in the leftmost column, then he can win the game. We will prove $\forall\ nP(n)$ by strong induction. We know that $P(1)$ is true, because the opponent is forced to take the poisoned cookie at his first turn.

Fix k $\geq$ 1 and assume that $P(j)$ is true for all $j \leq k$. We claim that $P(k+1)$ is true. It is the opponent's turn to move. If she picks the poisoned cookie, then the game is over and she loses. Otherwise, assume she picks the cookie in the top row in column j, or the cookie in the left column in row j , for some j with $2 \geq j \geq k+1$. The first player now picks the cookie in the left column in row j , or the cookie in the top row in column respectively. This leaves the position covered by $P(j-1)$ for his opponent, so by the inductive hypothesis, he can win.

■

3.4 Generalisations of Chomp

3.4.1 Three-dimensional Chomp

Three-dimensional Chomp has an initial chocolate bar of a cuboid of blocks indexed as (i, j, k). A move is to take a block together with any block all of whose indices are greater or equal to the corresponding index of the chosen block. In the same way Chomp can be generalised to any number of dimensions.

3.4.2 Numerical Version of Chomp

Chomp is sometimes described numerically. An initial natural number is given, and players alternate choosing positive divisors of the initial number, but may not choose 1 or a multiple of a previously chosen divisor. This game models n-dimensional Chomp, where the initial natural number has n prime factors and the dimensions of the Chomp board are given by the exponents of the primes in its prime factorization.

3.4.3 Ordinal Chomp

Ordinal Chomp is played on an infinite board with some of its dimensions ordinal numbers: for example a $2 \times (\omega + 4)$ bar. A move is to pick any block and remove all blocks with both indices greater than or equal the corresponding indices of the chosen block. The case of $\omega \times \omega \times \omega$ Chomp is a notable open problem; a $100 reward has been offered for finding a winning first move.

3.5 Open Question

Describe a winning strategy for that Chomp that applies for all rectangular grids of size $m \times n$ by describing the moves that the first player should follow.

4 The Angel Game

4.1 Game Description

The Angel and the Devil play their game on an infinite chessboard, with one square for each ordered pair of integers (x, y). On his turn, the Devil may eat any square of the board whatsoever; this square is then no longer available to the Angel. The Angel is a "chess piece" that can move to any uneaten square (X, Y) that is at most 1000 king's moves away from its present position (x, y) in other words, for which $|X - x|$ and $|Y - y|$ are at most 1000. Angels have wings, so that it does not matter if any intervening squares have already been eaten.

The Devil wins if he can strand the Angel, that is, surround him by a moat of eaten squares of width at least 1000. The Angel wins just if he can continue to move forever.

What we have described is more precisely called an *Angel of power* 1000. The *Angel Problem* is this:

Determine whether an Angel of some power can defeat the Devil.

Berlekamp showed that the Devil can beat an Angel of power one (a chess King) on any board of size at least 32 $\times$ 33. However, it seems that it is impossible for the Devil to beat a Knight, and that would imply that an Angel of power two (which is considerably stronger than a Knight) will win. Nobody has been able to prove it yet, even when we make it much easier by making the Angel have power 1000 or some larger number. The main difficulty seems to be that the Devil cannot ever make a mistake: once he's made some moves, no matter how foolish, he is in a strictly better position than he was at the start of the game, since the squares he's eaten can only hinder the Angel.

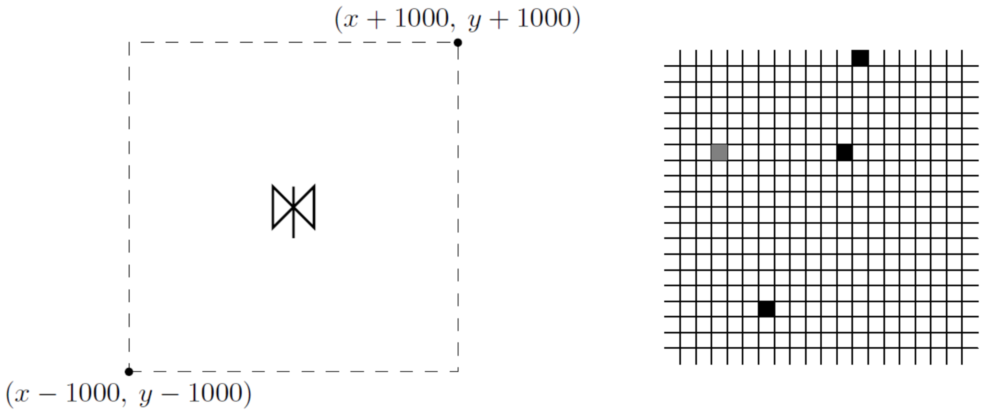

Figure 5 Left: An Angel about to fly; he can fly to any point in the square.
Right: A Devil eating a square; three have already been eaten.

4.2 Existence of a Winning Strategy

LEMMA: *The angel-devil game is determined. That is, either the angel or the devil has a winning strategy.*

PROOF: Assume the devil has no winning strategy. The angel can play as follows. In each turn he makes a move after which the devil does not have a winning strategy. By induction, such a move must always exist since otherwise the devil would have a winning strategy. The resulting angel strategy is obviously a winning strategy, simply because it allows the angel to play forever. ■

4.3 Fools Rush On Where Angels Fear to Tread

DEFINITION: A Fool is an Angel who is required always strictly to increase his y coordinate. So a Fool can make precisely those Angel moves from (x, y) to (X, Y) for which $Y > y$.

THEOREM: *The Devil can catch a Fool.*

PROOF: If the Fool is ever at some point P, he will be at all subsequent times in the "upward cone" from P, whose boundary is defined by the two upward rays of slope $\pm\frac{1}{1000}$ through P. Then we counsel the Devil to act as follows (Figure 6): he should truncate this cone by a horizontal line AB at a very large height H above the Fool's starting position, and use his first few moves to eat one out of every M squares along AB, where M is chosen so that this task will be comfortably finished when the Angel reaches a point Q on the halfway line that's distant $\frac{1}{2}H$ below AB (we'll arrange H to be exactly divisible by a large power of two).

At subsequent times, the Devil knows that the Fool will be safely ensconced in the smaller cone QCD, where CD is a sub interval of AB of exactly half its length, and for the next few of his moves, he should eat the second one of every M squares along the segment CD. He will have finished this by the time the Fool reaches a point R on the horizontal line $\frac{1}{4}H$ below AB. At later times, the Fool will be trapped inside a still smaller cone REF, with $EF = \frac{1}{2}CD = \frac{1}{2}AB$, and the Devil should proceed to eat the third one of every M squares along the segment EF of AB.

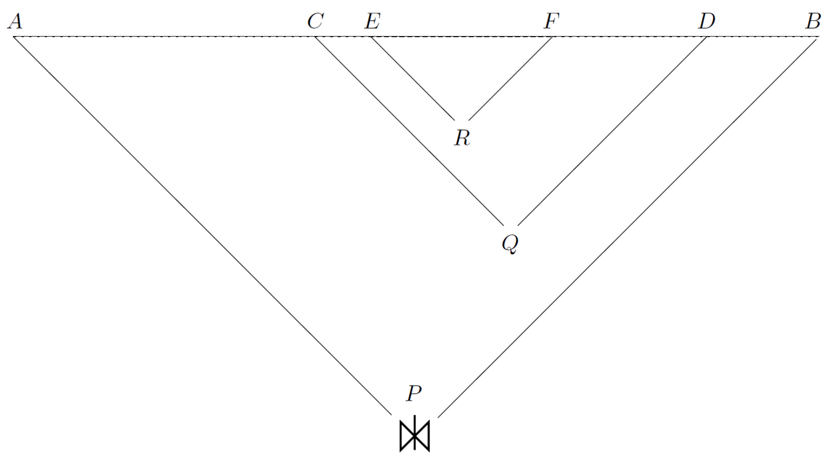

Figure 6 A Fool travelling north with the Devil eating along AB.

If he proceeds in this way, then by the time the Fool reaches the horizontal line at distance $H^{'} = 2^{-M}H$ below AB, the Devil will have eaten every square of the sub-segment of AB that might still be reached by the Fool. The Devil should then continue, by eating the first out of every M squares on the segment $A^{'}B^{'}$ just below this one, a task which will be finished before the Fool reaches the horizontal line distant $\frac{1}{2}H^{'}$ below $A^{'}B^{'}$ when he should start eating the second of every M squares on the portion $C^{'}D^{'}$ of $A^{'}B^{'}$ that is still accessible, and so on. We see that if we take H of the form 1000×2^{N}, where $N > 1000M$, then before the Fool crosses the horizontal line that is 1000 units below AB, the Devil will have eaten all squares between this line and AB that the Fool might reach, and so the Fool will be unable to move. ■

LEMMA: *If angel can survive arbitrarily long, then he can win.*

PROOF: Suppose the Angel has strategies for surviving arbitrarily long. At each turn, the Angel has only finitely many options. One of these options must still leave strategies for surviving arbitrarily long. Always take such a move $\rightarrow$ winning strategy for the Angel.

■

COROLLARY: *If the devil wins, then he only needs a finite board to win.*

■

4.4 Oddvar Kloster's proof

This proof presents a winning strategy for an Angel of power 2. This was proved by Kloster in 2006 for an *Angel of Power* 2.

Lemma: *An Angel of power 2 wins against the Devil.*

Proof: The Angel walks along the right of a directed path. Initially the path a vertical line, with the angel directly to the right of it.

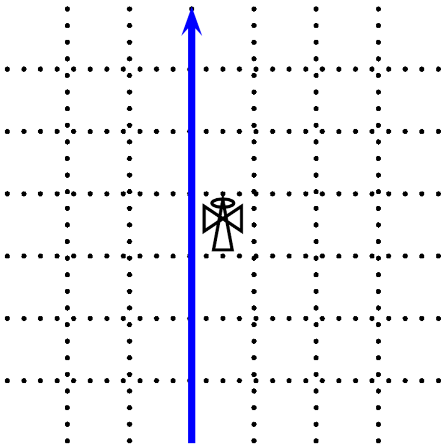

Figure 7 Path followed by the Angel

Each turn, the Angel tries to cover as much path as possible. The Angel moves along at least two segments of the path every time. If a right turn is made, the Angel makes a move of more than 2 segments.

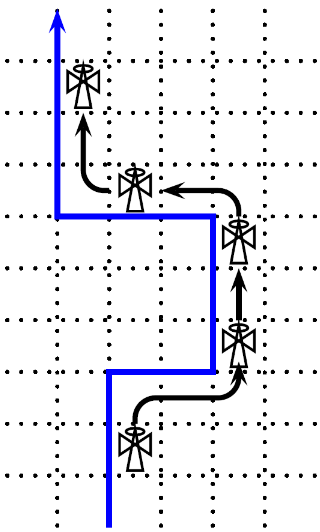

Figure 8 Angel moves at least two segments of the path each time

Definition: *Squares are either free □, or blocked ■. Squares to the left of the path are called evaded □. The evaded set is equal to the current path's left set. A blocked square is one that is in the current path's right set and has been eaten by the Devil. The uneaten squares in the path's right set are free.*

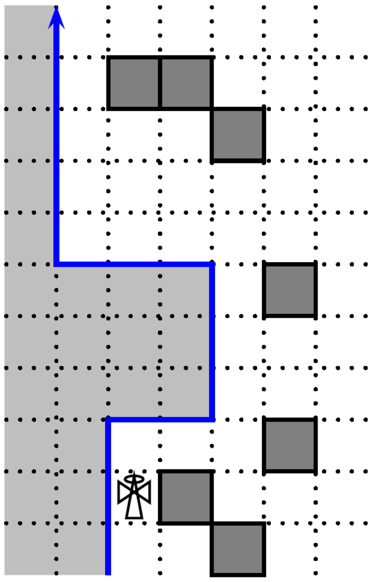

Figure 9 The 3 different types of squares on the board: free, blocked and evaded

Before moving, the Angel must adjust the path ahead, to avoid the Devil's traps. He is only allowed to move some section of the path to the right (w.r.t. the direction of the path) to evade some more squares. *The resulting path is called a **descendant** of the original path.*

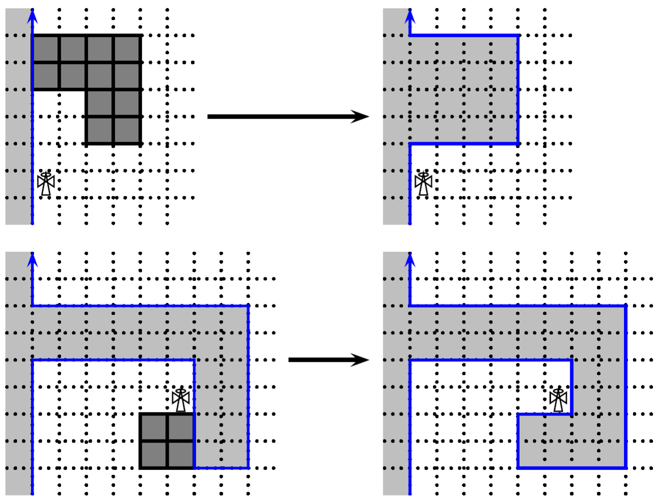

Figure 10 Descendant of the original path

The Angel needs a strategy that keeps the squares along the right of the path free and unevaded.

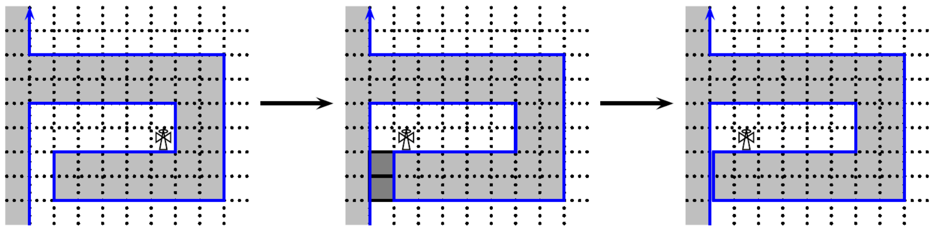

Figure 11 An Angel being trapped by the Devil

DEFINITION: *We assign an integer length L_k to a path k, namely the number of extra segments it has compared to the original path. We define $E_k(s)$ to be the number of squares, eaten before stage s, that are evaded by k. At stage s:*

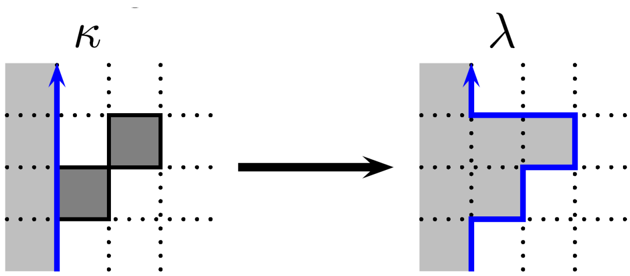

Figure 12 Illustration of path k and λ

QUESTION: ***How does the Angel adjust its path at stage s?***

ANSWER: Among all the descendants of the previous path, select the path k with maximal $2E_k(s) - L_k$. If there exists more than one such path, then select the one with maximal $E_k(s)$. ■

QUESTION: ***Can a square, just right of a future segment of the path, be evaded?***

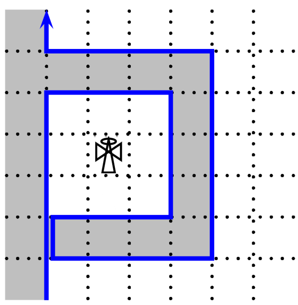

Figure 13 The strategy ensures that the Angel is not trapped in a situation like this

ANSWER: We will prove that the Angel, following our strategy, would have spotted this potential trap in time, and would have planned its path around the trap. ■

LEMMA: *An Angel of Power 2 by following the strategy of selecting the path with maximal $2E_k(s) - L_k$ will never get trapped.*

PROOF: Angel passes segments more than twice faster than the Devil can build:

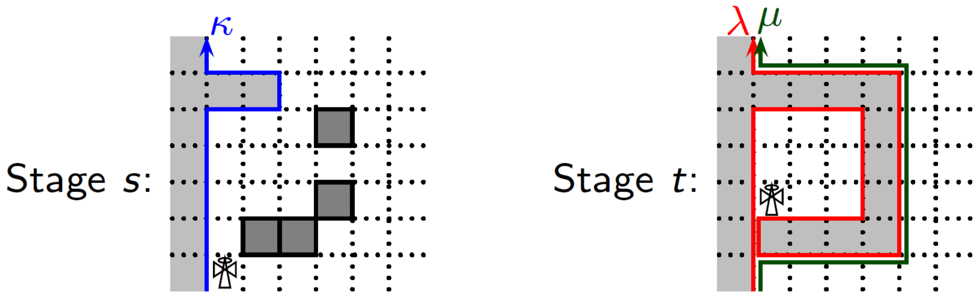

Figure 14 The Angel would have preferred path μ over path k.

$$L_\lambda - L_\mu > 2(E_\mu(t) - E_\mu(s))$$

Hence,

$$2E_\mu(s) - L_\mu > 2E_\mu(t) - L_\lambda$$

$$2E_\mu(s) - L_\mu \geq 2E_\lambda(t) - L_\lambda$$

$$2E_\mu(s) - L_\mu \geq 2E_k(s) - L_k$$

Hence, at stage s, the Angel would have preferred path μ over path k. ■

4.5 Open Question

***Can an Angel of power k win for some k?*.** This problem was proposed by John Conway and it is still an open problem. There is a \$100 prize for a winning Angel proof and \$1000 prize for a winning Devil proof.

5 Concluding Remarks

Thus we briefly went through two combinatorial games - Game of Chomps and Angel Game. There is a still an open question in both problems for the community. For the Game of Chomps, an explicit strategy for an $m \times n$ board is an open question. For the Angel Game, whether for a k-Angel, which one - the Devil or the Angel has a winning strategy is still an open question.

References

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6 https://en.wikipedia.org/wiki/Combinatorial_game_theory